

# An axiomatic look at a windmill

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**Abstract.** We present the problem stated in intuitive language as problem 2 at the 52nd International Mathematical Olympiad as a formal statement, and prove that it is valid in ordered regular incidence planes, the weakest ordered geometry whose models can be embedded in projective ordered planes.

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## 1. Introduction

Proposed by Geoffrey Smith of the University of Bath, the second problem on the first day of the 52nd IMO, held in Amsterdam, reads as follows (see [2] for the statements and proofs of all problems):

Let  $\mathcal{S}$  be a finite set of at least two points in the plane. Assume that no three points of  $\mathcal{S}$  are collinear. A *windmill* is a process that starts with a line  $l$  going through a single point  $P \in \mathcal{S}$ . The line rotates clockwise about the *pivot*  $P$  until the first time that the line meets some other point belonging to  $\mathcal{S}$ . This point,  $Q$ , takes over as the new pivot, and the line now rotates clockwise about  $Q$ , until it next meets a point of  $\mathcal{S}$ . This process continues indefinitely, with the pivot always being a point from  $\mathcal{S}$ .

Show that we can choose a point  $P$  in and a line  $l$  going through  $P$  such that the resulting windmill uses each point of  $\mathcal{S}$  as a pivot infinitely many times.

As stated, the problem appears on a first reading to be describing a process in Euclidean geometry or at any rate a geometry with a metric, as it appears

to require the existence of rotations, without really belonging to Euclidean geometry proper, given its strong combinatorial flavor.

The aim of this note is to state it as a theorem of ordered regular incidence planes.

We will proceed as follows: First, we will provide an axiom system for planar ordered domains (without the lower-dimension axiom), then we will state the windmill problem inside that formalism, and provide a proof for it. Next we will introduce axiomatically ordered regular incidence planes, state the windmill problem in that formalism, in which it turns out to be a universal statement (i.e. it does not contain any existential quantifier), and finally provide the rationale for the validity of the windmill problem inside that axiom system.

## 2. The axiomatic framework for planar ordered domains

The axiomatic framework is that of a very general two-dimensional theory of betweenness, the models of which will be referred to as *planar ordered domains* (see also [5]), axiomatized in terms of *points* as individual variables and the strict betweenness ternary predicate  $Z$ , with  $Z(abc)$  to be read as ‘ $b$  lies between  $a$  and  $c$ ’ (and the order is strict, i. e.  $b$  is different from  $a$  and  $c$ ), the axiom system consisting of the axioms A1-A5 axiomatizing  $\mathcal{L}$ , the universal theory of linear order (we omit throughout the paper universal quantifiers in universal sentences):

- A 1.**  $Z(abc) \rightarrow Z(cba)$ ,
- A 2.**  $Z(abc) \rightarrow \neg Z(acb)$ ,
- A 3.**  $Z(acb) \wedge Z(abd) \rightarrow Z(cbd)$ ,
- A 4.**  $Z(cab) \wedge Z(abd) \rightarrow Z(cbd)$ ,
- A 5.**  $c \neq d \wedge Z(abc) \wedge Z(abd) \rightarrow (Z(bcd) \vee Z(bdc))$ ,

the lower-dimension axiom, stating that there are three non-collinear points, and the Pasch axiom (here  $L$  stands for the collinearity predicate, defined by  $L(xyz) :\Leftrightarrow Z(xyz) \vee Z(yzx) \vee Z(zxy) \vee x = y \vee y = z \vee z = x$ ; although we do not have the concept of a ‘line’ in our language, we will refer to lines, saying that the point  $p$  lies on the line  $\langle a, b \rangle$ , for two distinct points  $a$  and  $b$ , if  $p = a$  or  $p = b$  or  $Z(apb)$  or  $Z(pba)$  or  $Z(bap)$ ):

- A 6.**  $(\forall abcde)(\exists f) [\neg L(abc) \wedge Z(adc) \wedge \neg L(ace) \wedge \neg L(edb) \rightarrow (Z(afb) \vee Z(bfc)) \wedge L(edf)]$ .

Notice that we do not ask the order to be dense or unending, and we also do not need the lower-dimension axiom, stating the existence of three non-collinear points.

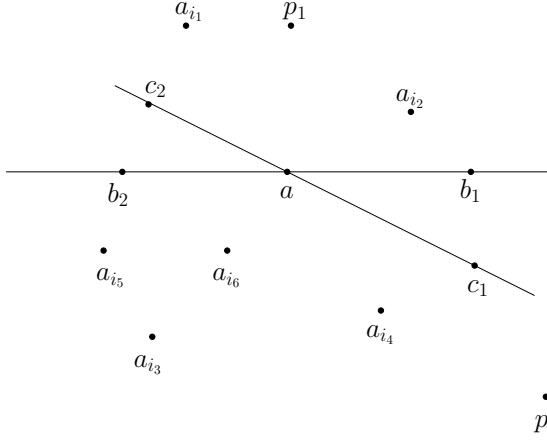


FIGURE 1.  $\pi_{1,1}^i(a, b_1, p_i, c_1)$ ,  $\pi_{1,0}^i(a, b_1, p_i, c_2)$ ,  $\pi_{0,1}^i(a, b_2, p_i, c_1)$ ,  $\pi_{0,0}^i(a, b_2, p_i, c_2)$

### 3. The windmill statement

Let  $r_m(k)$  denote the remainder of the division of  $k$  by  $m$ , and let  ${}^\epsilon\varphi$  stand for  $\varphi$  if  $\epsilon = 0$ , and for  $\neg\varphi$  if  $\epsilon = 1$ . Let  ${}_\epsilon(\varphi)$  stand for  $\perp$  (the absurdity sign) if  $\epsilon = 0$ , and for  $\varphi$  if  $\epsilon = 1$ . To help with the readability of the formal statement of the windmill problem, we introduce the following defined predicates (where  $j, k, l \in \{0, 1\}$ ):

$$\begin{aligned}
 \lambda(xyz) &\Leftrightarrow Z(xyz) \vee Z(yzx) \vee Z(zxy) \\
 \delta(abuv) &\Leftrightarrow (\exists t) \lambda(abt) \wedge Z(utv) \\
 \pi_{k,l}^j(a, b, p, c) &\Leftrightarrow ({}^{r_2(j+k+1)}\delta(acbp) \vee {}^{r_2(j+l)}(p = c)) \wedge {}^{r_2(j+l)}\delta(abc p) \\
 &\quad \wedge \bigwedge_{\{i | a_i \notin \{a, b, c, p\}\}} \\
 &\quad [({}^{(1-k)}\delta(abca_i) \wedge {}^l\delta(acba_i)) \vee ({}^k\delta(abca_i) \wedge {}^{(1-l)}\delta(acba_i))]
 \end{aligned}$$

Let  $\alpha(n) = n(n-1) + 1$ . Let  $K_n = \{f \mid f : \{1, 2, \dots, \alpha(n)\} \rightarrow \{1, 2, \dots, n\}, (\exists k(n)) k(n) \leq \alpha(n), \text{ such that the restriction of } f \text{ to } \{1, 2, \dots, k(n)\} \text{ is onto, } (\forall i) 3 \leq i \leq k(n), f(i) \neq f(i-1), f(i) \neq f(i-2), f(k(n)-1) = f(2), f(k(n)) = f(1)\}$ , let  ${}^A2$  stand for the set of all functions from  $A$  to  $\{0, 1\}$ , and in case  $A = \{1, 2, \dots, m\}$ , let us write  ${}^m2$  for  ${}^A2$ . For  $f \in K_n$ , we define  $f(0) = r_{n+1}(f(1) + f(2))$ .

With these definitions, we are ready to express the windmill theorem as the following statement:

$$\mathbf{WM} . \bigvee_{1 \leq i < j < k \leq n} L(a_i a_j a_k) \\ \bigvee_{f \in K_n, g \in k(n)2} \bigwedge_{i=3}^{k(n)} \pi_{1-g(i-1), g(i)}^{1-g(i-2)}(a_{f(i-1)}, a_{f(i-2)}, a_{f(i-3)}, a_{f(i)}).$$

To see that this is actually the windmill statement of Geoffrey Smith's problem, notice that  $\pi_{k,l}^j(a, b, p, c)$  means that  $c$  is the first point in the set  $\{c, a_1, \dots, a_n\} \setminus \{a, b\}$  the line determined by  $a$  and  $b$  meets, when it "rotates" around  $a$  "clockwise." The point  $p$  is there to fix the sense of the "rotation," and the sub- and superscripts  $j, k$ , and  $l$  are there to tell us whether  $p$  is in the half-plane toward which the ray  $\overrightarrow{ab}$  moves and whether  $c$  will be met by  $\overrightarrow{ab}$  or by the opposite ray during its "rotation" (see Fig. 1 for details). If we look at the "windmill" process of the original statement by Geoffrey Smith, we notice that the "windmill" consists of lines through single points of the given set  $S = \{a_1, \dots, a_n\}$ , and that pivots are obtained during what one could call "windmill stops." Instead of focusing on the "windmill" as composed by lines through single points of  $S$ , we decided to look at the "windmill" as a collection of "windmill stops," i. e. a collection of lines passing through exactly two points of  $S$ . The statement **WM** makes is that, given a set  $S$  of points  $a_1, \dots, a_n$ , that are such that no three are collinear, one can find a map  $f$  in  $K_n$ , such that the first windmill stop goes through  $a_{f(1)}$  and  $a_{f(2)}$  ( $a_{f(0)}$  being chosen as a point different from  $a_{f(1)}$  and  $a_{f(2)}$ , playing the role of  $p$  in the formula for  $\pi$ , i. e. determining the "clockwise" direction for the entire windmill process), with windmill pivots  $a_{f(i)}$ , with  $i \in \{1, \dots, k(n)\}$ , which, given the definition of  $K_n$ , exhaust  $S$ , and such that the last windmill stop has pivot  $a_{f(1)}$ , and goes through  $a_{f(1)}$  and  $a_{f(2)}$ , just like the stop we started with. During the whole process, the point  $p$  determining the orientation changes (whenever  $c$  becomes pivot, and the windmill stop is the line  $ac$ , the new point  $p$  is chosen to be the point  $b$  from the windmill stop  $ab$  that "rotated" to stop at  $c$ , i. e. the  $b$  for which  $\pi_{k,l}^j(a, b, p, c)$  holds), but the orientation itself, and thus the "clockwise" sense of the "rotation", stays the same. The reason we cannot stay with the same  $p$  is that the windmill can stop at  $p$ , i. e. we can have  $c = p$  in  $\pi_{k,l}^j(a, b, p, c)$ .

## 4. The proof

Since the proof in [2] is carried out inside the Euclidean plane, we need to provide a variant thereof that would hold inside planar ordered domains. We treat the case in which  $n$ , the number of points in the set  $S$ , is even and the case in which it is odd separately.

If  $n$  is even, then let  $a_s$  be any vertex of the convex hull of  $S$  (these elementary notions of convex geometry can all be defined and have the usual properties, as shown in [1]). Among the lines  $\langle a_s, a_i \rangle$  with  $i \in \{1, \dots, n\} \setminus \{s\}$  there must be one having an equal number of points on each of its sides. We denote that particular value of  $i$  by  $t$ , and start the windmill process with the first pivot

in  $a_s$  and windmill stop at  $a_s a_t$ , i. e.  $f(1) = s$ ,  $f(2) = t$ . The first value of  $p$  will be  $a_{r_{n+1}(f(1)+f(2))}$  (choosing the index to be  $r_{n+1}(f(1) + f(2))$  had no other function than making sure it is different from  $f(1)$  and  $f(2)$ ), and we'll choose  $g(1) = 1$ ,  $g(2) = 0$ . We define the direction  $a_{f(1)} \overrightarrow{a_{f(2)}}$  as the *East* (and thus  $a_{f(2)} \overrightarrow{a_{f(1)}}$  as the *West*, the half-plane determined by  $\langle a_{f(1)}, a_{f(2)} \rangle$  in which  $a_{f(0)}$  lies as the *Southern* half-plane. These directions change in the course of the windmill process as follows: if  $\pi_{k,l}^j(a, b, p, c)$ , then  $\overrightarrow{ab}$  and  $\overrightarrow{ac}$  point in the same "direction" (i. e. both East or both West) if  $k = l$  and in different directions if  $k \neq l$ , whereas the half-plane determined by  $ab$  in which  $p$  lies has the same name as the half-plane determined by  $ac$  in which  $b$  lies if  $k = j$ , and the opposite name if  $k \neq j$ . The Southern half-plane determined by a windmill stop  $ab$  will be denoted by  $\sigma_{ab}$ , and the Northern half-plane by  $\nu_{ab}$

We now look at the possible changes in the difference  $\delta(ab) = N(ab) - S(ab)$  between the number  $N(ab)$  of points in  $\nu_{ab}$  and the number  $S(ab)$  of points in  $\sigma_{ab}$  during the windmill process. We want to show that  $\delta$  can take only the values 0 and 2.

The next stop will be a point  $a_{f(3)}$ , with the property that there is no point in  $S$  between the rays  $a_{f(1)} \overrightarrow{a_{f(2)}}$  and  $a_{f(1)} \overrightarrow{a_{f(3)}}$ . Given that  $a_{f(1)}$  is a vertex of the convex hull of  $S$ , and that the sense of the "rotation about  $a_{f(1)}$ " is "clock-wise", i. e. towards the Southern half-plane,  $a_{f(3)}$  must lie in  $\sigma_{a_{f(1)} a_{f(2)}}$ , thus  $\delta(a_{f(1)} a_{f(3)})$  must be 2, as the Northern half-plane gains a point, namely  $a_{f(2)}$ , and the Southern half-plane loses one, namely  $a_{f(3)}$ .

Point  $a_{f(3)}$  becomes a pivot during the next stage of the windmill process, and at the next stop, i. e. when line  $\langle a_{f(1)}, a_{f(3)} \rangle$  "rotates about  $a_{f(3)}$ " into  $\langle a_{f(3)}, a_{f(4)} \rangle$  (where  $a_{f(4)}$  is the point in  $S$  which lies either in  $\nu(a_{f(3)} a_{f(1)})$  and for which there is no point in  $S$  between  $a_{f(3)} \overrightarrow{a_{f(1)}}$  and  $a_{f(3)} \overrightarrow{a_{f(4)}}$ , nor between the rays opposite to the above two, or else lies in  $\sigma(a_{f(3)} a_{f(1)})$ , and no point in  $S$  lies between the ray opposite to  $a_{f(3)} \overrightarrow{a_{f(4)}}$  and  $a_{f(3)} \overrightarrow{a_{f(1)}}$ , nor between the rays opposite to the above two), point  $a_{f(1)}$  will be in  $\sigma_{a_{f(3)} a_{f(4)}}$ , and we distinguish two cases: (i)  $a_{f(4)}$  is in  $\sigma_{a_{f(1)} a_{f(3)}}$  and (ii)  $a_{f(4)}$  is in  $\nu_{a_{f(1)} a_{f(3)}}$ . In case (i),  $\sigma_{a_{f(3)} a_{f(4)}}$  both gains and loses a point when compared to  $\sigma_{a_{f(1)} a_{f(3)}}$ , and thus  $\delta$  stays the same, namely 2, and we are back to the situation we were in at the windmill stop  $a_{f(1)} a_{f(3)}$ , with the pivot,  $a_{f(4)}$ , East of the other point,  $a_{f(3)}$ , of the windmill stop, with  $\delta$  taking the value 2.

In case (ii),  $\sigma_{a_{f(3)} a_{f(4)}}$  will have one point,  $a_{f(1)}$ , more than  $\sigma_{a_{f(1)} a_{f(3)}}$ , and  $\nu_{a_{f(3)} a_{f(4)}}$  will have one point,  $a_{f(4)}$ , less than  $\nu_{a_{f(1)} a_{f(3)}}$ , so  $\delta$  will become 0 for the windmill stop  $a_{f(3)} a_{f(4)}$ , and we are back to a configuration of the type we started with at windmill stop  $a_{f(1)} a_{f(2)}$ , with the pivot,  $a_{f(4)}$ , to the West of the other point of the windmill stop,  $a_{f(3)}$ . However, there is one difference:  $a_{f(4)}$  no longer needs to be a vertex of the convex hull of  $S$ , and so

we can no longer say, as in the case of the windmill stop  $a_{f(1)}a_{f(2)}$ , that the next point the windmill will meet, while “rotating about  $a_{f(4)}$ ”, will lie in  $\sigma_{a_{f(4)}a_{f(3)}}$ . It can lie in either the Northern and the Southern half-plane, and so a new situation can appear, that in which  $a_{f(5)}$  lies in  $\nu_{a_{f(3)}a_{f(4)}}$ . There is thus, one situations that still needs to be analyzed in complete generality: the pivot  $a$  is West of  $b$  at the windmill stop  $ab$ , with  $\delta(ab) = 0$ . The case left open is that in which  $c$ , the first point line  $\langle a, b \rangle$  meets while “rotating about  $a$ ”, lies in  $\nu_{ab}$ . In that case  $\nu_{ac}$  both gains and loses a point with respect to  $\nu_{ab}$  (it gains  $b$  and loses  $c$ ), so we are back into the previous situation, i. e. the pivot  $c$  lies to the West of the other point,  $a$ , of the windmill stop and  $\delta(ac) = 0$ .

We conclude that, during the entire windmill process, we have only the following two possible situations for the windmill stop  $ab$ : either  $\delta(ab) = 0$ , and the pivot  $a$  lies to the West of  $b$ , or else  $\delta(ab) = 2$ , and the pivot  $a$  lies to the East of  $b$ .

At every stage of the windmill process, the *imprint* of the “East” on the convex hull of  $S$ , i. e. the point where the Eastward pointing ray of that windmill stop intersects the convex hull of  $S$ , moves in clockwise direction towards  $a_{f(1)}$ , except when the pivot is a vertex of the convex hull of  $S$ , in which case the imprint stays put for that one step in the process, but will have to move on in the next, as the pivot changes at that step. After a finite number of steps, in fact in no more than  $n(n-1)/2$  steps (given that this is the total number of lines that can be formed by joining two points in  $S$ , and thus the upper bound on the number of windmill stops), the imprint will be for the last time in  $\sigma_{a_{f(1)}a_{f(2)}}$ , in the sense that at the next windmill stop the imprint of the “East” will have to be either  $a_{f(1)}$  or be in  $\nu_{a_{f(1)}a_{f(2)}}$ . However, it cannot be in  $\nu_{a_{f(1)}a_{f(2)}}$ , for  $a_{f(1)}$  would have lied between the two windmill stops, contradicting the definition of  $\pi$ .

The pivot  $a$  of the last windmill stop  $\langle a, b \rangle$  for which the Eastward imprint is in  $\sigma_{a_{f(1)}a_{f(2)}}$  cannot lie in  $\sigma_{a_{f(1)}a_{f(2)}}$ , for if it did, then  $\langle a, a_{f(1)} \rangle$  would be the next windmill stop, and  $\delta(aa_{f(1)})$  would have to be  $\leq -1$ , contradicting the fact that  $\delta$  takes only non-negative values.

If  $a$  lies on  $\langle a_{f(1)}, a_{f(2)} \rangle$ , then  $a$  would have to be  $a_{f(2)}$ , and, since the next windmill stop, after  $\langle a_{f(2)}, b \rangle$  is  $\langle a_{f(1)}, a_{f(2)} \rangle$ , we would be back in the starting position, with  $\langle a_{f(1)}, a_{f(2)} \rangle$  as windmill stop and  $a_{f(1)}$  as pivot, and thus we’d be done.

If  $a$  lies in  $\nu_{a_{f(1)}a_{f(2)}}$ , then, since  $\delta(aa_{f(1)})$  would have to be positive, and since 2 is the only positive value it is allowed to take, there can be no point in  $S$  lying between the rays  $a_{f(1)}\overrightarrow{a_{f(2)}}$  and  $a_{f(1)}\overrightarrow{a}$ , so the next windmill stop, after  $\langle a_{f(1)}, a \rangle$ , has to be  $\langle a_{f(1)}, a_{f(2)} \rangle$ . In that case, we are not quite back where we started from, for although the windmill stop is  $\langle a_{f(1)}, a_{f(2)} \rangle$ , the pivot is  $a_{f(2)}$ , not  $a_{f(1)}$  the way it was at the start. However, by the same reasoning that showed us that, in case we start with a line which has the

same number of points on each of its sides as first windmill stop, we arrive in at most  $n(n-1)/2$  steps back to itself, we conclude that we'll be either back to  $\langle a_{f(1)}, a_{f(2)} \rangle$  with  $a_{f(2)}$  as pivot, in which case starting with  $\langle a_{f(1)}, a_{f(2)} \rangle$  as windmill stop and with  $a_{f(2)}$  as pivot, we arrive back to the same windmill stop and pivot in at most  $n(n-1)/2$  steps, or else we'll be back to  $\langle a_{f(1)}, a_{f(2)} \rangle$  with  $a_{f(1)}$  as pivot, in which case starting with  $\langle a_{f(1)}, a_{f(2)} \rangle$  as windmill stop and with  $a_{f(1)}$  as pivot, we arrive back to the same windmill stop and pivot in at most  $n(n-1)$  steps. That each point in  $S$  must have become a pivot by the time the windmill stop returns for the first time to  $\langle a_{f(1)}, a_{f(2)} \rangle$  is easily seen by noticing that, at this stage  $\nu(a_{f(1)}a_{f(2)})$  has become what used to be  $\sigma(a_{f(1)}a_{f(2)})$  at the start of the process, and that a point in  $S$  can move from the Southern to the Northern half-plane only by having been touched by a windmill stop.

In case  $n$  is odd, say  $n = 2k + 1$ , we let  $a_s$  be any vertex of the convex hull of  $S$  and choose among the lines  $\langle a_s, a_i \rangle$  with  $i \in \{1, \dots, n\} \setminus \{s\}$  the one having  $k + 1$  points on one side and  $k$  points on the other side. Just like in the case in which  $n$  is even, we denote that particular value of  $i$  by  $t$ , and start the windmill process with the first pivot in  $a_s$  and windmill stop at  $\langle a_s, a_t \rangle$ , i. e.  $f(1) = s$ ,  $f(2) = t$ . We now distinguish two possibilities: the first value of  $p$  chosen to start the windmill process, i. e.  $a_{r_{n+1}(f(1)+f(2))}$ , can be (i) in the half-plane with  $k$  points or (ii) in the half-plane with  $k + 1$  points. In case (i), we notice, as in the  $n$  even case treated earlier, that during the windmill process  $\delta$  can take on only two values, namely 1 and 3. If we follow the path of the imprint of the West on the convex hull of  $S$ , we notice that it moves in the “clockwise” direction (i. e. moving inside  $\nu_{a_{f(1)}a_{f(2)}}$ ) at every step of the process, unless the pivot is a vertex of the convex hull of  $S$ , in which case it rests for one step of the windmill process, but will have to move afterwards. After a finite number of steps, no more than twice the total number of lines that can be formed by two points in  $S$  (since each such line has two directions that can become the “West” direction during the windmill process), the imprint of the West has to come back to its original location,  $a_{f(1)}$  (it cannot jump over it, the reason being the same as in the  $n$  already discussed even case). Now, the other point  $a$  of the windmill stop  $\langle a, a_{f(1)} \rangle$  we arrive at, when the imprint of the West is back at  $a_{f(1)}$  must be  $a_{f(2)}$ . To see this, notice that, if  $a$  were in  $\sigma_{a_{f(1)}a_{f(2)}}$ , then  $\delta(aa_{f(1)})$  would have to be  $\geq 3$ , so at the next windmill stop  $\langle a_{f(1)}, b \rangle$ , we'd have  $\delta(a_{f(1)}b) \geq 5$ , which is not possible. If  $a$  were in  $\nu_{a_{f(1)}a_{f(2)}}$ , then  $\delta(aa_{f(1)})$  would have to be  $\leq -1$ , which is impossible, and thus  $a = a_{f(2)}$ . Case (ii) is treated analogously, by noticing that throughout the windmill process  $\delta$  takes on only the values  $-1$  and 1. For reasons similar to those mentioned in the  $n$  even case, each point in  $S$  must have become a pivot during the windmill process.

## 5. Validity in ordered regular incidence planes

There is a weaker axiom system, for *ordered regular incidence planes* from which **WM** can be derived. It cannot be expressed in terms of *points* and  $Z$ , as it is based on the notion of *sides* of a line in a plane, put forward by Sperner in [6], from which  $Z$  can be defined, but which cannot, in general, be defined in terms of  $Z$ . It can be expressed in a two-sorted language, with variables for *points* (to be represented by lower-case Latin characters) and for *lines* (to be represented by lower-case Gothic characters), with two relation symbols,  $I$ , with  $I(a\mathfrak{g})$  to be read as ‘point  $a$  is incident with line  $\mathfrak{g}$ ’, and  $D$ , with  $D(agh)$  to be read as ‘the points  $a$  and  $b$  lie on different sides of line  $\mathfrak{g}$ ’. With  $\delta(ab\mathfrak{g}h) :\Leftrightarrow [(D(agh) \wedge D(bgh)) \vee (\neg D(agh) \wedge \neg D(bgh))]$  and  ${}^\epsilon\delta$  standing for  $\delta$  if  $\epsilon = 1$  and for  $\neg\delta$  if  $\epsilon = 0$ , the axioms are (see [3]):

- J 1.**  $(\forall ab)(\exists {}^1\mathfrak{g}) a \neq b \rightarrow I(a\mathfrak{g}) \wedge I(b\mathfrak{g}),$
- J 2.**  $(\forall \mathfrak{g})(\exists a_1 a_2 a_3 a_4) \bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge \bigwedge_{i=1}^4 I(a_i \mathfrak{g})$
- J 3.**  $(\exists abc)(\forall \mathfrak{g}) \neg(I(a\mathfrak{g}) \wedge I(b\mathfrak{g}) \wedge I(c\mathfrak{g})),$
- J 4.**  $D(agh) \rightarrow \neg I(a\mathfrak{g}),$
- J 5.**  $D(agh) \rightarrow D(bga),$
- J 6.**  $\neg I(c\mathfrak{g}) \wedge D(agh) \rightarrow (D(agc) \vee D(bgc)),$
- J 7.**  $\neg(D(agh) \wedge D(bgc) \wedge D(cga)),$
- J 8.**  $[\bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge h_i \neq h_j \wedge \bigwedge_{i=1}^4 I(a_i h_i) \wedge h_i \neq \mathfrak{g} \wedge$   
 $((\bigwedge_{i=1}^4 I(a_i \mathfrak{g})) \vee (\bigwedge_{i=1}^4 I(oh_i)))]$   
 $\rightarrow [\bigvee_{\substack{\epsilon_i \in \{0,1\} \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 2}} {}^{\epsilon_1}\delta(a_3 a_4 h_1 h_2) \wedge {}^{\epsilon_2}\delta(a_2 a_4 h_1 h_3) \wedge {}^{\epsilon_3}\delta(a_2 a_3 h_1 h_4)].$

J6 is a weak variant of Pasch’s axiom, stating that if a line  $\mathfrak{g}$  does not pass through any of the points  $a, b$ , and  $c$ , and  $a$  and  $b$  are on different sides of  $\mathfrak{g}$  then so are at least one of the pairs  $\{a, c\}$  and  $\{b, c\}$ . J7 is a variant of Pasch’s theorem, stating that a line cannot separate all three pairs  $\{a, b\}$ ,  $\{b, c\}$ , and  $\{c, a\}$ . One of its special cases, when  $a = b = c$ , implies that  $a$  and  $b$  can be on different sides of  $\mathfrak{g}$  only if  $a \neq b$ . That these versions are called “weak” stems from the fact that, if a line  $\mathfrak{g}$  separates the points  $a$  and  $b$ , it no longer means that there is a point on  $\mathfrak{g}$  which is between  $a$  and  $b$ . Indeed, the line  $\mathfrak{g}$  and the line determined by  $a$  and  $b$  may have no point in common (a simple example is provided by the submodel of the ordered affine plane over  $\mathbb{Q}$  whose points have coordinates whose denominators are powers of 2, with the plane separation relation inherited from the ordered affine plane over  $\mathbb{Q}$ ; see [4] for other examples). The meaning of J8 is best understood in terms of the notion of *separation*  $//$  (with  $ab//cd$  to be read as the point-pair  $(a, b)$  separates the point-pair  $(c, d)$ ), defined by



$$\begin{aligned}
a_1 a_2 // a_3 a_4 \quad :\Leftrightarrow \quad & (\exists \mathfrak{g} \mathfrak{h} \mathfrak{k}) \bigwedge_{i=1}^4 I(a_i \mathfrak{g}) \wedge \bigwedge_{1 \leq i < j \leq 4} a_i \neq a_j \wedge I(a_1 \mathfrak{h}) \wedge I(a_2 \mathfrak{k}) \\
& \wedge \mathfrak{h} \neq \mathfrak{g} \wedge \mathfrak{k} \neq \mathfrak{g} \wedge \neg \delta(a_3 a_4 \mathfrak{h} \mathfrak{k}).
\end{aligned} \tag{5.1}$$

One part of it (corresponding to the  $\bigwedge_{i=1}^4 I(a_i \mathfrak{g})$  disjunct) states that, if  $a_1, a_2, a_3, a_4$  are four different collinear points, then exactly one of the separation relations  $a_1 a_2 // a_3 a_4$ ,  $a_1 a_3 // a_2 a_4$ ,  $a_1 a_4 // a_2 a_3$  holds. Its other part (corresponding to the  $\bigwedge_{i=1}^4 I(o \mathfrak{h}_i)$  disjunct) is the dual statement (in the sense of projective geometry).

Joussen [3] showed that any model  $\mathfrak{M}$  of J1-J8 can be embedded in a projective ordered plane  $\mathfrak{P}$ , whose separation relation  $//_{\mathfrak{P}}$  is an extension of the separation relation  $//_{\mathfrak{M}}$ , defined in  $\mathfrak{M}$  terms of  $I_{\mathfrak{M}}$  and  $D_{\mathfrak{M}}$  by (5.1).

The windmill statement **WM** remains the same, if we change the definition of the defined notions  $L$  and  $\delta$  occurring in it to:

$$\begin{aligned}
\delta(abuv) & \Leftrightarrow (\exists \mathfrak{g}) a \neq b \wedge I(a \mathfrak{g}) \wedge I(b \mathfrak{g}) \wedge D(u \mathfrak{g} v), \\
L(abc) & \Leftrightarrow (\exists \mathfrak{g}) (I(a \mathfrak{g}) \wedge I(b \mathfrak{g}) \wedge I(c \mathfrak{g})) \vee a = b \vee b = c \vee c = a.
\end{aligned}$$

To see that **WM** is true in ordered regular incidence planes, suppose **WM** were not derivable from J1-J8. Then there would have to exist a model  $\mathfrak{M}$  of J1-J8 in which **WM** is false, i. e. in which  $\neg \mathbf{WM}$  holds. Now notice that  $\neg \mathbf{WM}$  can be expressed as an existential statement in the following way:

$$\begin{aligned}
& (\exists a_i)_{1 \leq i \leq n} (\exists \mathfrak{g}_{ij})_{1 \leq i < j \leq n} \bigwedge_{1 \leq i < j \leq n} a_i \neq a_j \wedge \bigwedge_{1 \leq i < j < k \leq n} I(a_i \mathfrak{g}_{ij}) \wedge I(a_j \mathfrak{g}_{ij}) \\
& \wedge \bigwedge_{(i,j) \neq (k,l), 1 \leq i < j \leq n, 1 \leq k < l \leq n} \mathfrak{g}_{ij} \neq \mathfrak{g}_{kl} \\
& \wedge \bigwedge_{f \in K_n, g \in {}^{k(n)}2} \left[ \bigvee_{i=3}^{k(n)} \neg \pi_{1-g(i-1), g(i)}^{1-g(i-2)} (a_{f(i-1)}, a_{f(i-2)}, a_{f(i-3)}, a_{f(i)}) \right].
\end{aligned}$$

in which the  $\delta(a_i a_j uv)$  occurring in the  $\pi$ 's are just  $D(u \mathfrak{g}_{ij} v)$ .

Then  $\neg \mathbf{WM}$ , as an existential statement, would have to hold in the ordered projective plane  $\mathfrak{P}$ , in which  $\mathfrak{M}$  can be embedded, as well. If we remove from the projective plane  $\mathfrak{P}$  a line which does not contain any of the points  $a_i$  which  $\neg \mathbf{WM}$  claims to exist in  $\mathfrak{M}$ , such that the windmill process does not close regardless of the choice of its starting position, we obtain a model  $\mathfrak{N}$  of A1-A6 in which  $\neg \mathbf{WM}$  holds, a contradiction.

**Theorem 1.**  $\{J1-J2, J4-J8\} \vdash \mathbf{WM}$ , where **WM** is expressed in terms of points, lines,  $I$ , and  $D$ .

By defining  $Z$  in terms of  $I$  and  $D$  by

$$Z(abc) :\Leftrightarrow (\exists \mathfrak{g}\mathfrak{h}) \mathfrak{h} \neq \mathfrak{g} \wedge I(a\mathfrak{g}) \wedge I(b\mathfrak{g}) \wedge I(c\mathfrak{g}) \wedge I(b\mathfrak{h}) \wedge D(a\mathfrak{h}c), \quad (5.2)$$

one can compare the set of  $Z$ -consequences of the axiom system  $\{J1\text{-}J2, J4\text{-}J8\}$  to  $\{A1\text{-}A6\}$ . It turns out that the  $Z$  defined by (5.2) satisfies  $A1\text{-}A5$ , but does not need to satisfy  $A6$ . On the other hand,  $J2$  does not follow from  $\{A1\text{-}A6\}$ . Although formally incomparable, intuitively  $\{J1\text{-}J2, J4\text{-}J8\}$  is the weaker axiom system.

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